The Wave Motion of a Revolving Shaft, and a Suggestion as to the Angular Momentum in a Beam of Circularly Polarised Light.

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When a shaft of circular section is revolving uniformly, and is transmitting power uniformly, a row of particles originally in a line parallel to the axis will lie in a spiral of constant pitch, and the position of the shaft at any instant may be described by the position of this spiral.

Let us suppose that the power is transmitted from left to right, and that as viewed from the left the revolution is clockwise. Then the spiral is a left-handed screw. Let it be on the surface, and there make an angle ϵ with the axis. Let the radius of the shaft be a, and let one turn of the spiral have length λ along the axis. We may term λ the wave-length of the spiral. We have $\tan\epsilon = 2\pi a/\lambda$. If the orientation of the section at the origin at time t is given by $\theta = 2\pi \mathrm{N}t$, where N is the number of revolutions per second, the orientation of the section at x is given by

$$\theta = 2\pi \mathrm{N}t - \frac{x}{a}\tan\epsilon = \frac{2\pi}{\lambda}(\mathrm{N}\lambda t - x), \tag{1}$$

which means movement of orientation from left to right with velocity $\mathrm{N}\lambda$.

The equation of motion for twist waves on a shaft of circular section is

$$\frac{d^2\theta}{dt^2} = \mathrm{U}_n{}^2 \frac{d^2\theta}{dx^2}, \tag{2}$$

where $\mathrm{U}_n{}^2$ = modulus of rigidity/density = n/ρ.

Though (1) satisfies (2), it can hardly be termed a solution for $d^2\theta/dt^2$, and $d^2\theta/dx^2$ in (1) are both zero. But we may adapt a solution of (2) to fit (1) if we assume certain conditions in (1).

The periodic value

$$\theta = \Theta \sin \frac{2\pi}{l}(\mathrm{U}_n t - x)$$

satisfies (2), and is a wave motion with velocity U_n and wave-length l. Make l so great that for any time or for any distance under observation $\mathrm{U}_n t/l$ and x/l are so small that the angle may be put for the sine. Then

$$\theta = \Theta \frac{2\pi}{l}(\mathrm{U}_n t - x). \tag{3}$$

This is uniform rotation. It means that we only deal with the part of

the wave near a node, and that we make the wave-length l so great that for a long distance the "displacement curve" obtained by plotting θ against t coincides with the tangent at the node. We must distinguish, of course, between the wave-length l of the periodic motion and the wave-length λ of the spiral.

We can only make (1) coincide with (3) by putting

$$\Theta/l = 1/\lambda \quad \text{and} \quad \mathrm{N}\lambda = \mathrm{U}_n.$$

Then it follows that for a given value of N, the impressed speed of uniform rotation, there is only one value of λ or one value of ϵ for which the motion may be regarded as part of a natural wave system, transmitted by the elastic forces of the material with velocity $= \sqrt{(n/\rho)}$. There is therefore only one "natural" rate of transmission of energy.

The value of ϵ is given by

$$\tan\epsilon = 2\pi a/\lambda = 2\pi a\mathrm{N}/\mathrm{N}\lambda = 2\pi a\mathrm{N}/\mathrm{U}_n = 2\pi a\mathrm{N}\sqrt{(\rho/n)}.$$

Suppose, for instance, that a steel shaft with radius $a = 2$ cm., density $\rho = 7{\cdot}8$, and rigidity $n = 10^{12}$ is making $\mathrm{N} = 10$ revs. per sec. We may put $\tan\epsilon = \epsilon$, since it is very small. The shaft is twisted through 2π in length λ or through $2\pi/\lambda$ per centimetre, and the torque across a section is

$$\mathrm{G} = \tfrac{1}{2}\, n\pi a^4\, 2\,\pi/\lambda = n\pi^2 a^4 \mathrm{N}\sqrt{(\rho/n)},$$

since

$$\lambda = \frac{\mathrm{U}_n}{\mathrm{N}} = \frac{1}{\mathrm{N}}\sqrt{\frac{n}{\rho}}.$$

The energy transmitted per second is

$$2\pi\mathrm{NG} = 2\pi^3 a^4 \mathrm{N}^2\sqrt{(n\rho)}.$$

Putting 1 H.P. $= 746 \times 10^7$ ergs per second, this gives about 38 H.P.

But a shaft revolving with given speed N can transmit any power, subject to the limitation that the strain is not too great for the material. When the power is not that "naturally" transmitted, we must regard the waves as "forced." The velocity of transmission is no longer U_n, and forces will have to be applied from outside in addition to the internal elastic forces to give the new velocity.

Let H be the couple applied per unit length from outside. Then the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = \mathrm{U}_n{}^2\frac{d^2\theta}{dx^2} + \frac{2\mathrm{H}}{\pi a^4},$$

where $\tfrac{1}{2}\pi a^4$ is the moment of inertia of the cross section. Assuming that the condition travels on with velocity U unchanged in form,

$$\frac{d\theta}{dt} = -\mathrm{U}\frac{d\theta}{dx} \quad \text{and} \quad \mathrm{H} = \tfrac{1}{2}\pi a^4(\mathrm{U}^2 - \mathrm{U}_n{}^2)\frac{d^2\theta}{dx^2},$$

or H has only to be applied where $d^2\theta/dx^2$ has value, that is where the twist is changing.

The following adaptation of Rankine's tube method of obtaining wave velocities* gives these results in a more direct manner. Suppose that the shaft is indefinitely extended both ways. Any twist disturbance may be propagated unchanged in form with any velocity we choose to assign, if we apply from outside the distribution of torque which, added to the torque due to strain, will make the change in twist required by the given wave motion travelling at the assigned speed.

Let the velocity of propagation be U from left to right, and let the displacement at any section be θ, positive if clockwise when seen from the left. The twist per unit length is

$$\frac{d\theta}{dx} = -\frac{1}{\mathrm{U}}\frac{d\theta}{dt} = -\frac{\dot{\theta}}{\mathrm{U}}.$$

The torque across a section from left to right in clockwise direction is

$$-\tfrac{1}{2}\,n\pi a^4\frac{d\theta}{dx} = \frac{n\pi a^4}{2\mathrm{U}}\,.\,\dot{\theta}.$$

Let the shaft be moved from right to left with velocity U; then the disturbance is fixed in space, and if we imagine two fixed planes drawn perpendicular to the axis, one, A, at a point where the disturbance is θ and the other, B, outside the wave system, where there is no disturbance, the condition between A and B remains constant, except that the matter undergoing that condition is changing. Hence the total angular momentum between A and B is constant. But no angular momentum enters at B, since the shaft is there untwisted and has merely linear motion. At A, then, there must be on the whole no transfer of angular momentum from right to left. Now, angular momentum is transferred in three ways:—

1. By the carriage by rotating matter. The angular momentum per unit length is $\frac{1}{2}\rho\pi a^4\dot{\theta}$, and since length U per second passes out at A, it carries out $\frac{1}{2}\rho\pi a^4\dot{\theta}\mathrm{U}$.

2. By the torque exerted by matter on the right of A on matter on the left of A. This takes out $-n\pi a^4\dot{\theta}/2\mathrm{U}$.

3. By the stream of angular momentum by which we may represent the forces applied from outside to make the velocity U instead of U_n.

If H is the couple applied per unit length, we may regard it as due to the flow of angular momentum L along the shaft from left to right, such that $\mathrm{H} = -d\mathrm{L}/dx$. There is then angular momentum L flowing out per second from right to left. Since the total flow due to (1), (2), and (3) is zero,

$$\tfrac{1}{2}\rho\pi a^4\dot{\theta}\mathrm{U} - n\pi a^4\dot{\theta}/2\mathrm{U} - \mathrm{L} = 0,$$

* 'Phil. Trans.,' 1870, p. 277.

and $$L = \frac{\pi a^4 \dot{\theta}}{2}\left(\rho U - \frac{n}{U}\right) = \frac{\rho \pi a^4 \dot{\theta}}{2U}(U^2 - U_n^2) = -\frac{\rho \pi a^4}{2}\frac{d\theta}{dx}(U^2 - U_n^2),$$

and $$H = -\frac{dL}{dx} = \frac{\rho \pi a^4}{2}\frac{d^2\theta}{dx^2}(U^2 - U_n^2).$$

If $H = 0$, either $U^2 = U_n^2$ when the velocity has its "natural value," or $d^2\theta/dx^2 = 0$, and the shaft is revolving with uniform twist in the part considered.

Now put on to the system a velocity U from left to right. The motion of the shaft parallel to its axis is reduced to zero, and the disturbance and the system H will travel on from left to right with velocity U. A "forced" velocity does not imply *transfer* of physical conditions by the material with that velocity. We can only regard the conditions as reproduced at successive points by the aid of external forces. We may illustrate this point by considering the incidence of a wave against a surface. If the angle of incidence is i and the velocity of the wave is V, the line of contact moves over the surface with velocity $v = V/\sin i$, which may have any value from V to infinity. The velocity v is not that of transmission by the material of the surface, but merely the velocity of a condition impressed on the surface from outside.

Probably in all cases of transmission with forced velocity, and certainly in the case here considered, the velocity depends upon the wave-length, and there is dispersion.

With a shaft revolving N times per second $U = N\lambda$, and it is interesting to note that the group velocity $U - \lambda dU/d\lambda$ is zero. It is not at once evident what the group velocity signifies in the case of uniform rotation. In ordinary cases it is the velocity of travel of the "beat" pattern, formed by two trains of slightly different frequencies. The complete "beat" pattern is contained between two successive points of agreement of phase of the two trains. In our case of superposition of two strain spirals with constant speed of rotation, points of agreement of phase are points of intersection of the two spirals. At such points the phases are the same, or one has gained on the other by 2π. Evidently as the shaft revolves these points remain in the same cross-section, and the group velocity is zero.

With deep water waves the group velocity is half the wave velocity, and the energy flow is half that required for the onward march of the waves.* The energy flow thus suffices for the onward march of the group, and the case suggests a simple relation between energy flow and group velocity.

But the simplicity is special to unforced trains of waves. Obviously,

* O. Reynolds, 'Nature,' August 23, 1877; Lord Rayleigh, 'Theory of Sound,' vol. 1, p. 477.

it does not hold when there are auxiliary working forces adding or subtracting energy along the waves. For the revolving shaft the simple relation would give us no energy flow, whereas the strain existing in the shaft implies transmission of energy at a rate given as follows.

The twist per unit length is $d\theta/dx$, and therefore the torque across a section is $-\frac{1}{2}n\pi a^4 d\theta/dx$, or $\frac{1}{2}n\pi a^4\dot{\theta}/\mathrm{U}$, since $d\theta/dx = -\dot{\theta}/\mathrm{U}$. The rate of working or of energy flow across the section is $\frac{1}{2}n\pi a^4\dot{\theta}^2/\mathrm{U}$.

The relation of this to the strain and kinetic energy in the shaft is easily found. The strain energy per unit length being $\frac{1}{2}$ (couple $\times$ twist per unit length) is $\frac{1}{4}n\pi a^4(d\theta/dx)^2$, which is $\frac{1}{4}n\pi a^4\dot{\theta}^2/\mathrm{U}^2$. The kinetic energy per unit length is $\frac{1}{2}\rho\pi a^4\dot{\theta}^2$, or, putting $\rho = n/\mathrm{U}_n^2$, is $\frac{1}{4}n\pi a^4\dot{\theta}^2/\mathrm{U}_n^2$.

In the case of natural velocity, for which no working forces along the shaft are needed, when $\mathrm{U}=\mathrm{U}_n=\sqrt{(n/\rho)}$, the kinetic energy is equal to the strain energy at every point and the energy transmitted across a section per second is that contained in length U_n.

But if the velocity is forced this is no longer true,* and it is easily shown that the energy transferred is that in length $\dfrac{2\mathrm{U}}{1+\mathrm{U}^2/\mathrm{U}_n^2}$, which is less than U if $\mathrm{U}>\mathrm{U}_n$, and is greater than U if $\mathrm{U}<\mathrm{U}_n$.

It appears possible that always the energy is transmitted along the shaft at the speed U_n. If the forced velocity $\mathrm{U}>\mathrm{U}_n$, we may, perhaps, regard the system in a special sense as a natural system with a uniform rotation superposed on it.

Let us suppose that the whole of the strain energy in length U_n is transferred per second while only the fraction μ of the kinetic energy is transferred, the fraction $1-\mu$ being stationary.

The energy transferred : strain energy in U_n : kinetic energy in U_n $= 1/\mathrm{U} : \mathrm{U}_n/2\mathrm{U}^2 : \mathrm{U}_n/2\mathrm{U}_n^2$.

Put $\mathrm{U} = p\mathrm{U}_n$, and our supposition gives

$$\frac{1}{p\mathrm{U}_n} = \frac{1}{2p^2\mathrm{U}_n}+\frac{\mu}{2\mathrm{U}_n} \quad \text{or} \quad \mu = \frac{2}{p}-\frac{1}{p^2} = 1-\left(1-\frac{1}{p}\right)^2.$$

If the forced velocity $\mathrm{U}<\mathrm{U}_n$, we may regard the system as a natural one, with a uniform stationary strain superposed on it.

We now suppose that the whole of the kinetic energy is transferred, but only a fraction ν of the strain energy, and we obtain

$$\frac{1}{p\mathrm{U}_n} = \frac{\nu}{2p^2\mathrm{U}_n}+\frac{1}{2\mathrm{U}_n} \quad \text{or} \quad \nu = 2p-p^2 = 1-(1-p)^2.$$

* In the Sellmeier model illustrating the dispersion of light, the particles may be regarded as outside the material transmitting the waves and as applying forces to the material which make the velocity forced. The simple relation between energy flow and group velocity probably does not hold for this model.

It is perhaps worthy of note that a uniform longitudinal flow of fluid may be conceived as a case of wave motion in a manner similar to that of the uniform rotation of a shaft.

A Suggestion as to the Angular Momentum in a Beam of Circularly Polarised Light.

A uniformly revolving shaft serves as a mechanical model of a beam of circularly polarised light. The expression for the orientation θ of any section of the shaft distant x from the origin, $\theta = 2\pi\lambda^{-1}(\mathrm{U}t - x)$, serves also as an expression for the orientation of the disturbance, whatever its nature, constituting circularly polarised light.

For simplicity, take a shaft consisting of a thin cylindrical tube. Let the radius be a, the cross-section of the material s, the rigidity n, and the density ρ. Let the tube make N revolutions per second, and let it have such twist on it that the velocity of transmission of the spiral indicating the twist is the natural velocity $\mathrm{U}_n = \sqrt{(n/\rho)}$.

Repeating for this special case what we have found above, the strain energy per unit length is $\frac{1}{2}n\epsilon^2 s$, or, since $\epsilon = a d\theta/dx = -a\dot{\theta}/\mathrm{U}_n$, the strain energy is $\frac{1}{2}na^2 s\dot{\theta}^2/\mathrm{U}_n^2 = \frac{1}{2}\rho a^2 s\dot{\theta}^2$.

But the kinetic energy per unit length is also $\frac{1}{2}\rho a^2 s\dot{\theta}^2$; so that the total energy in length U_n is $\rho a^2 s\dot{\theta}^2\mathrm{U}_n$. The rate of working across a section is

$$n\epsilon s a\dot{\theta} = na^2 s\dot{\theta}^2/\mathrm{U}_n = \rho a^2 s\dot{\theta}^2\mathrm{U}_n,$$

or the energy transferred across a section is the energy contained in length U_n.

If we put E for the energy in unit volume and G for the torque per unit area, we have

$$\mathrm{G}s\dot{\theta} = \mathrm{E}s\mathrm{U}_n,$$

whence

$$\mathrm{G} = \mathrm{EU}_n/\dot{\theta} = \mathrm{EN}\lambda/2\pi\mathrm{N} = \mathrm{E}\lambda/2\pi.$$

The analogy between circularly polarised light and the mechanical model suggests that a similar relation between torque and energy may hold in a beam of such light incident normally on an absorbing surface. If so, a beam of wave-length λ containing energy E per unit volume will give up angular momentum $\mathrm{E}\lambda/2\pi$ per second per unit area. But in the case of light waves $\mathrm{E} = \mathrm{P}$, where P is the pressure exerted. We may therefore put the angular momentum delivered to unit area per second as

$$\mathrm{P}\lambda/2\pi.$$

In the 'Philosophical Magazine,' 1905, vol. 9, p. 397, I attempted to show that the analogy between distortional waves and light waves is still closer, in that distortional waves also exert a pressure equal to the energy per unit volume. But as I have shown in a paper on "Pressure Perpendicular to the

Shear Planes in Finite Pure Shears, etc." (*ante*, p. 546), the attempt was faulty, and a more correct treatment of the subject only shows that there is probably a pressure. We cannot say more as to its magnitude than that if it exists it is of the order of the energy per unit volume.

When a beam is travelling through a material medium we may, perhaps, account for the angular momentum in it by the following considerations. On the electromagnetic theory the disturbance at any given point in a circularly polarised beam is a constant electric strain or displacement f uniformly revolving with angular velocity $\dot{\theta}$. In time dt it changes its direction by $d\theta$.

This may be effected by the addition of a tangential strain $fd\theta$; or the rotation is produced by the addition of tangential strain $f\dot{\theta}$ per second, or by a current $f\dot{\theta}$ along the circle described by the end of f. We may imagine that this is due to electrons drawn out from their position of equilibrium so as to give f, and then whirled round in a circle so as to give a circular convection current $f\dot{\theta}$. Such a circular current of electrons should possess angular momentum.

Let us digress for a moment to consider an ordinary conduction circuit as illustrating the possession of angular momentum on this theory. Let the circuit have radius a and cross-section s, and let there be N negative electrons per unit volume, each with charge e and mass m, and let these be moving round the circuit with velocity v. If i is the total current, $i = \mathrm{N}sve$. The angular momentum will be

$$\mathrm{N}s2\pi a \,.\, mva = 2\pi a^2 im/e = 2\mathrm{A}im/e = 2\mathrm{M}m/e,$$

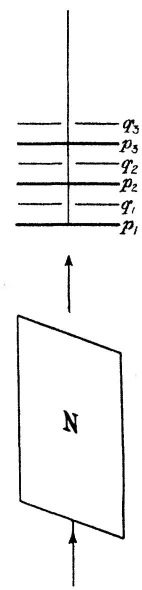
where A is the area of the circuit and M is the magnetic moment. This is of the order of $2\mathrm{M}/10^7$.

It is easily seen that this result will hold for any circuit, whatever its form if A is the projection of the circuit on a plane perpendicular to the axis round which the moment is taken and if $\mathrm{M} = \mathrm{A}i$. If we suppose that a current of negative electrons flows round the circuit in this way and that the reaction while their momentum is being established is on the material of the conductor, then at make of current there should be an impulse on the conductor of moment $2\mathrm{M}/10^7$. If the circuit could be suspended so that it lay in a horizontal plane and was able to turn about a vertical axis in a space free from any magnetic field, we might be able to detect such impulse if it exists. But it is practically impossible to get a space free from magnetic intensity. If the field is H, the couple in the circuit due to it is proportional to HM. It would require exceedingly careful construction and adjustment of the circuit to ensure that the component of the couple due to the field

about the vertical axis was so small that its effect should not mask the effect of the impulsive couple. The electrostatic forces, too, might have to be considered as serious disturbers.

Returning to a beam of circularly polarised light, supposed to contain electrons revolving in circular orbits in fixed periodic times, the relations between energy and angular momentum are exactly the same as those in a revolving shaft or tube, and the angular momentum transmitted per second per square centimetre is $E\lambda/2\pi = P\lambda/2\pi$, where P is the pressure of the light per square centimetre on an absorbing surface.

The value of this in any practical case is very small. In light pressure experiments, P is detected by the couple on a small disc, of area A say, at an arm b and suspended by a fibre. What we observe is the moment APb. If the same disc is suspended by a vertical fibre attached at its centre and the same beam circularly polarised in both cases is incident normally upon it, according to the value suggested the torque is $AP\lambda/2\pi$.



The ratio of the two is $\lambda/2\pi b$. Now b is usually of the order of 1 cm. Put $\lambda = 6 \times 10^{-5}$, or, say, $2\pi/10^{-5}$, and the ratio becomes 10^{-5}.

It is by no means easy to measure the torque APb accurately, and it appears almost hopeless to detect one of a hundred-thousandth of the amount. The effect of the smaller torque might be multiplied to some extent, as shown in accompanying diagram.

Let a series of quarter wave plates, $p_1, p_2, p_3, \ldots$, be suspended by a fibre above a Nicol prism N, through which a beam of light is transmitted upwards, and intermediate between these let a series of quarter wave plates, $q_1, q_2, q_3, \ldots$, be fixed, each with a central hole for the free passage of the fibre. The beam emerges from N plane polarised. If N is placed so that the beam after passing through p_1 is circularly polarised, it has gained angular momentum, and therefore tends to twist p_1 round. The next plate q_1 is to be arranged so that the beam emerges from it plane polarised and in the original plane. It then passes through p_2, which is similar to p_1, and again it is circularly polarised and so exercises another torque. The process is repeated with q_2 and p_3, and so on till the beam is exhausted. By revolving N through a right angle round the beam, the effect is reversed. But, even with such multiplications, my present experience of light forces does not give me much hope that the effect could be detected, if it has the value suggested by the mechanical model.